

The Luminosity Distance in Perturbed FLRW Spacetimes

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Submitted: 9-July-2003

ABSTRACT

We derive an expression for the luminosity distance in FLRW spacetimes affected by scalar perturbations. Our expression is complete to linear order and is expressed entirely in terms of standard cosmological parameters and observational quantities. We illustrate the result by calculating the RMS scatter in the usual luminosity distance in flat $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$ and $(0.3, 0.7)$ cosmologies. In both cases the scatter is appreciable at high redshifts, and rises above 11% at $z = 2$, where it may be the dominant noise term in the Hubble diagram based on SN Ia.

Key words: Distance scale; gravitation; gravitational lensing.

1 INTRODUCTION

A substantial body of modern theoretical cosmology is concerned with the theory of small metric perturbations about Friedman-Lemaître-Robertson-Walker (FLRW) cosmologies. In particular, cosmologists often derive exact formulae for observables in such spacetimes and then apply standard statistical techniques in recognition of the stochastic nature of the perturbations. A great benefit of such a programme is that correlations between effects previously treated as distinct may be used to illuminate cosmological questions.

As astronomers and cosmologists work with ever more distant sources, it becomes increasingly important to understand and account for the deviations of our Universe from the simple FLRW spacetimes. One example, among many, where such an understanding is important is in the use of type Ia supernovae as standard candles to infer luminosity distance as a function of source redshift, and hence to demonstrate that the Universe is currently in a phase of accelerated expansion (Reiss et al. 1998; Perlmutter et al. 1999). The inference that the Universe is accelerating was made on the basis of the luminosity distance formula appropriate for the background cosmology, not to the perturbed cosmology in which we live. The gravitational effects of large-scale structure were taken to contribute to the error budget at a level determined by the numerical studies of Wambsganss et al. (1997). Gravitational lensing by large-scale structure is known to produce effects on the order of those found in these numerical studies, and this is usually assumed to be the dominant effect on the luminosity distance (Kantowski, Vaughan & Branch 1995; Holz & Wald 1998;

Holz 1998; Kaiser 1998). However, so far as we know, no rigorous exploration of this assumption has been attempted. In any case, the division of the effects of large-scale structure into categories such as lensing, time-delays, and others is largely artificial: we measure luminosities and redshifts in the real Universe and no idealized FLRW background exists.

In this paper we investigate the luminosity distance in linearly perturbed FLRW spacetimes. Our results are obtained by direct integration of the geodesic equations. A complementary approach has been employed by Sasaki (1987). Our results improve on those of Sasaki in two ways. First, we present explicit formulae for curved cases, whereas Sasaki's solutions in these cases require a prior integration of the perturbed geodesic equations. Second, our formulae are functions only of standard cosmological parameters and observational quantities.

The plan of this paper is as follows. In section 2 we derive a formula for the luminosity distance, accurate to first order, in metric perturbed FLRW spacetimes. In section 3 we evaluate the cosmological weak lensing correction to the usual luminosity distance as a function of redshift in two model flat spacetimes, with $\Omega_m = 1$ and $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$. Section 4 summarizes our results.

In this paper we use units such that $G = c = 1$. Greek indices μ, ν, \dots run over $\{0, 1, 2, 3\}$, and Roman indices i, j, \dots run over $\{1, 2, 3\}$. The spacetime metric is taken to have signature $+2$, and the Riemann and Ricci tensor conventions are given by $[\nabla_\alpha, \nabla_\beta]v^\mu = R^\mu{}_{\nu\alpha\beta}v^\nu$ and $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$.

2 ANALYSIS

In any spacetime there is a relation between the element of cross-sectional area of a radiation source, dA , and the solid angle that this source is observed to subtend at an observer, $d\Omega$. Although dA is unobservable, this relation is usually used to define a distance measure r , the angular-diameter distance, according to $dA = r^2 d\Omega$. In a general spacetime r will depend upon the particular locations of the source and observer, and possibly also on the choice of multiple null paths along which the source is observed. In perfectly homogeneous and isotropic FLRW spacetimes r depends only upon the observed redshift of the source and the cosmological parameters describing the evolution of the scale factor: the density parameters in matter, curvature, and cosmological constant and the Hubble constant, Ω_m , Ω_κ , Ω_Λ and H_0 . In any spacetime, the luminosity distance, d_L , is related to r and the observed redshift z of the source according to $d_L = (1+z)^2 r$. In this section we exploit our previous solution (Pyne & Birkinshaw 1996) for r in linearly perturbed FLRW spacetimes to find d_L .

We choose the cosmological model to be an FLRW spacetime with embedded small scalar perturbations ϕ (the extension to vector and tensor perturbations is conceptually straightforward). The metric of such a spacetime may be written in the form

$$ds^2 = a^2 \left[-(1+2\phi)d\eta^2 + (1-2\phi)\gamma^{-2} (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \right] \quad (1)$$

where $\gamma = 1 + \kappa \hat{r}^2/4$, κ is the spatial curvature parameter (± 1 or 0), and $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$.¹

Consider the radiation emitted by a source at event \mathcal{E} , corresponding to conformal time η_e , that reaches an observer at event \mathcal{O} at conformal time η_o at the spatial origin. Let the four-velocities of the source and observer be written $u_e^\mu = (1/a_e)(1 - \phi_e, v_e^i)$ and $u_o^\mu = (1/a_o)(1 - \phi_o, v_o^i)$ respectively.² Suppose, further, that the observer chooses a spatial frame such that the source appears in the \hat{z} -direction on the observer's sky. Then the relation between the cross-sectional area of the source at \mathcal{E} , dA , and the solid angle the source subtends at \mathcal{O} , $d\Omega$, was shown in Pyne & Birkinshaw (1996) to be

$$dA = d\Omega \left(1 + 2v_o^{\hat{z}} \right) \frac{a_e^2 \hat{r}_e^2}{\gamma_e^2} (1 - 2\phi_e) \det M \quad (2)$$

where

$$\begin{aligned} \det M = & 1 \\ & + \frac{4}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta (\eta - \eta_e) \phi_{,\eta} \\ & - \frac{4}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta \sin_\kappa(\eta - \eta_e) \gamma \phi_{,\hat{z}} \\ & + \frac{2}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta \sin_\kappa(\eta - \eta_e) \hat{r} \gamma [\phi_{,\hat{x}\hat{x}} + \phi_{,\hat{y}\hat{y}}] \end{aligned} \quad (3)$$

the integrals being taken along the path $x^{(0)} = (\eta, 0, 0, \hat{r}(\eta))$,

which is a null geodesic of the background spacetime, with affine parameter η , provided that $\hat{r} = 2 \tan_\kappa((\eta_o - \eta)/2)$. On this path $\gamma = \sec_\kappa^2((\eta_o - \eta)/2)$. In (3) and the following, the subscript κ on a trigonometric function denotes a set of three functions: for $\kappa = 1$ the trigonometric function itself, for $\kappa = -1$ the corresponding hyperbolic function, and for $\kappa = 0$ the first term in the series expansion of the function.³

The terms in (2) have the following interpretations:

$$\begin{aligned} (1 + 2v_o^{\hat{z}}) &= \left(\begin{array}{c} \text{special relativistic transformation} \\ \text{of solid angle at } \mathcal{O} \text{ due to} \\ \text{observer's local peculiar velocity} \end{array} \right) \\ \frac{a_e^2 \hat{r}_e^2}{\gamma_e^2} (1 - 2\phi_e) &= \left(\begin{array}{c} \text{metric factors in induced} \\ \text{area two-form in} \\ \text{rest frame of emitter} \end{array} \right) \\ \det M &= \left(\begin{array}{c} \text{effect of the gravitational field} \\ \text{on the light rays as they travel} \\ \text{from source to observer} \end{array} \right) \end{aligned}$$

In particular, the first two integrals in expression (3) for $\det M$ describe longitudinal effects of the perturbations, while the last term describes the transverse effects usually interpreted as gravitational lensing.

It is convenient to write (3) in the form

$$\begin{aligned} \det M = & 1 - 4\phi_o \\ & + \frac{4}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta (\eta - \eta_e) \phi_{,\eta} \\ & - \frac{4}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta \sin_\kappa(\eta - \eta_e) \phi_{,\eta} \\ & - \frac{4}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta \cos_\kappa(\eta - \eta_e) \phi \\ & + \frac{2}{\sin_\kappa(\eta_o - \eta_e)} \int_{\eta_o}^{\eta_e} d\eta \sin_\kappa(\eta - \eta_e) \hat{r} \gamma [\phi_{,\hat{x}\hat{x}} + \phi_{,\hat{y}\hat{y}}] \end{aligned} \quad (4)$$

which is achieved by noting that the directional derivative along $x^{(0)}$, $d\phi/d\eta = \phi_{,\eta} - \gamma \phi_{,\hat{z}}$, and performing an integration by parts.

While (2) is, in principle, a complete description of the angular-diameter distance in an FLRW spacetime perturbed by scalar perturbations, it suffers from a number of defects that limit its practical utility for observational cosmologists. Most seriously, it exhibits explicit dependence on the unobservable point of emission \mathcal{E} rather than on the observable redshift. Less seriously, it does not explicitly depend on the usual cosmological parameters. We proceed to remedy both defects.

Let \mathcal{Q} at conformal time η_q and at spatial position $(0, 0, \hat{r}(\eta_q))$ on $x^{(0)}$ be an event with the property that the redshift of a source at \mathcal{Q} observed at \mathcal{O} and computed in the background metric is numerically the same as that of a source at \mathcal{E} observed at \mathcal{O} computed in the full perturbed spacetime. Since η_e and η_q will differ only at first order, and (4) is valid only to this order, we can replace η_e by η_q in (4) without further modification. The metric two-form factors are trickier, however. Pyne & Birkinshaw (1996) showed how to calculate the coordinate displacements between \mathcal{E} and \mathcal{Q}

¹ A hat is used merely to distinguish this particular coordinate system, and not to denote a normalized vector.

² Note that these velocities are properly normalized if their spatial components are treated as first-order quantities.

³ Pyne & Birkinshaw (1996) write, more precisely, λ_e where we write η_e , but the two quantities differ only at first order, which renders them equivalent in (2).

suffered by a photon due to the presence of the perturbations, $(\delta\eta, \delta\hat{x}^i)$. The components of the displacement needed here are those corresponding to conformal time and longitudinal coordinate \hat{z} . They are

$$\delta\eta = \frac{a_q}{\dot{a}_q} (v_q^{\hat{z}} \gamma_e - v_o^{\hat{z}} \gamma_o + \phi_q - \phi_o + k_q^{(1)0} - k_o^{(1)0}) \quad (5)$$

$$\delta\hat{z} = -\gamma_q (\delta\eta + I_S) \quad (6)$$

where we have used a dot to denote a conformal-time derivative. In the above, the perturbation to the time-like component of the photon wavevector

$$k^{(1)0} = -2\phi + 2 \int_{\eta_o}^{\eta} d\eta \phi_{,\eta} \quad (7)$$

and

$$I_S = 2 \int_{\eta_o}^{\eta_q} d\eta \phi \quad (8)$$

is the usual expression for the Shapiro delay, both expressions being taken over the background path $x^{(0)}$.⁴ We point out that the displacement $\delta\eta$ is easily interpreted as the sum of three effects: a differential doppler shift, a differential gravitational redshift, and what is commonly termed the “integrated Sachs-Wolfe effect” in studies of the CMBR.

The expressions (5,6) allow us to Taylor expand the metric two-form factors about their values at \mathcal{Q} with the result⁵

$$\begin{aligned} \frac{a_e^2 \hat{r}_e^2}{\gamma_e^2} &\approx \frac{a_q^2 \hat{r}_q^2}{\gamma_q^2} + \left(\frac{a^2 \hat{r}^2}{\gamma^2} \right)_{,\eta} \delta\eta + \left(\frac{a^2 \hat{r}^2}{\gamma^2} \right)_{,\hat{z}} \delta\hat{z} \\ &= \frac{a_q^2 \hat{r}_q^2}{\gamma_q^2} + 2 \frac{\dot{a}_q}{a_q} \left(\frac{a_q^2 \hat{r}_q^2}{\gamma_q^2} \right) \delta\eta \\ &\quad + 2 \left(\frac{a_q^2 \hat{r}_q^2}{\gamma_q^2} \right) \frac{\cos\kappa(\eta_o - \eta_q)}{\hat{r}_q} \delta\hat{z} \end{aligned} \quad (9)$$

Recognizing that along the background path $\hat{r}/\gamma = \sin\kappa(\eta_o - \eta)$ we see

$$\frac{a_e^2 \hat{r}_e^2}{\gamma_e^2} \approx \frac{a_q^2 \hat{r}_q^2}{\gamma_q^2} \left[1 + 2 \frac{\dot{a}_q}{a_q} \delta\eta - 2 \cot\kappa(\eta_o - \eta_q) \delta\eta - 2 \cot\kappa(\eta_o - \eta_q) I_S \right] \quad (10)$$

Since $(1+z)^2 a_q \hat{r}_q / \gamma_q = d_L^{(0)}$, the luminosity distance of the background cosmological model, equations (2), (4) and (10) lead to

$$d_L = d_L^{(0)} (1 + \delta) \quad (11)$$

with

$$\begin{aligned} \delta &= v_o^{\hat{z}} + \frac{\dot{a}_q}{a_q} \delta\eta - \cot\kappa(\eta_o - \eta_q) \delta\eta \\ &\quad - \cot\kappa(\eta_o - \eta_q) I_S - \phi_q - 2\phi_o \end{aligned}$$

⁴ In the notation of Pyne & Birkinshaw (1996), $\delta\eta = \delta\lambda_q + x^{(1)0}(\lambda_e)$ and $\delta x^i = x^{(0)i}(\lambda_e) - x^{(0)i}(\lambda_q) + x^{(1)i}(\lambda_e)$. By Taylor expansion, $\delta x^i \approx k_q^{(0)i} \delta\lambda_q + x^{(1)i}(\lambda_e)$. An integration by parts on the expression for $x^{(1)i}(\lambda_e)$ given in Pyne & Birkinshaw then produces (5,6).

⁵ We note that the transverse derivatives in the Taylor expansion do not contribute once they are evaluated at \mathcal{Q} .

$$\begin{aligned} &+ \frac{2}{\sin\kappa(\eta_o - \eta_q)} \int_{\eta_o}^{\eta_q} d\eta (\eta - \eta_q) \phi_{,\eta} \\ &- \frac{2}{\sin\kappa(\eta_o - \eta_q)} \int_{\eta_o}^{\eta_q} d\eta \sin\kappa(\eta - \eta_q) \phi_{,\eta} \\ &- \frac{2}{\sin\kappa(\eta_o - \eta_q)} \int_{\eta_o}^{\eta_q} d\eta \cos\kappa(\eta - \eta_q) \phi \\ &+ \frac{1}{\sin\kappa(\eta_o - \eta_q)} \int_{\eta_o}^{\eta_q} d\eta \sin\kappa(\eta - \eta_q) \hat{r} \gamma [\phi_{,\hat{x}\hat{x}} + \phi_{,\hat{y}\hat{y}}] \end{aligned} \quad (12)$$

We now show how each term on the right-hand side of the above equation may be expressed in terms of observable quantities and standard cosmological parameters. First we change to the more familiar comoving coordinates (η, x, y, z) in which the background metric assumes the form

$$a^2(\eta) \left(d\eta + \frac{dr^2}{\sqrt{1 - \kappa r^2}} + r^2 d\theta^2 + r^2 \sin\theta d\phi^2 \right) \quad (13)$$

Here $r = \sqrt{x^2 + y^2 + z^2}$ is the comoving radial coordinate distance from the origin.⁶ This change is effected by the transformation $x^i = \hat{x}^i / \gamma$. A number of useful results can now be derived. First, along $x^{(0)}$, $r = \sin\kappa(\eta_o - \eta)$ and, in particular, $\sin\kappa(\eta_o - \eta_q) = r_q$, the comoving coordinate distance to \mathcal{Q} . From this follows $\cos\kappa(\eta_o - \eta) = \sqrt{1 - \kappa r^2}$ and the change in measure, $dr = -\cos\kappa(\eta_o - \eta) d\eta$.

On $x^{(0)}$, the Jacobian of the coordinate change may be shown to take the value

$$\frac{\partial x^i}{\partial \hat{x}^j} = \begin{pmatrix} \frac{1}{\gamma} & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & \frac{1}{\gamma} - \frac{\kappa}{2} \frac{\hat{z}^2}{\gamma^2} \end{pmatrix} \quad (14)$$

which secures, again on $x^{(0)}$,

$$\phi_{,xx} = \gamma^2 \phi_{\hat{x}\hat{x}} \quad (15)$$

$$\phi_{,yy} = \gamma^2 \phi_{\hat{y}\hat{y}} \quad (16)$$

Finally, it will be convenient to employ the common formulae for trigonometric functions of angular sums, in the forms

$$\begin{aligned} \sin\kappa(\eta - \eta_q) &= \sin\kappa(\eta - \eta_o) \cos\kappa(\eta_o - \eta_q) \\ &\quad + \cos\kappa(\eta - \eta_o) \sin\kappa(\eta_o - \eta_q) \\ &= -r \sqrt{1 - \kappa r_q^2} + r_q \sqrt{1 - \kappa r^2} \end{aligned} \quad (17)$$

$$\begin{aligned} \cos\kappa(\eta - \eta_q) &= \cos\kappa(\eta - \eta_o) \cos\kappa(\eta_o - \eta_q) \\ &\quad - \kappa \sin\kappa(\eta - \eta_o) \sin\kappa(\eta_o - \eta_q) \\ &= \sqrt{1 - \kappa r_q^2} \sqrt{1 - \kappa r^2} - \kappa r r_q \end{aligned} \quad (18)$$

$$\begin{aligned} \eta - \eta_q &= -(\eta_o - \eta) + (\eta_o - \eta_q) \\ &= -\sin\kappa^{-1} r + \sin\kappa^{-1} r_q \end{aligned} \quad (19)$$

Using these identities, we find

$$\begin{aligned} \delta &= (v_q^{\hat{z}} - \phi_o - 2\phi_q + I_{ISW}) \\ &\quad - r_q^{-1} \sqrt{1 - \kappa r_q^2} \delta\eta - r_q^{-1} \sqrt{1 - \kappa r_q^2} I_S \\ &\quad - 2r_q^{-1} \int_0^{r_q} \frac{dr}{\sqrt{1 - \kappa r^2}} \left[-\sin\kappa^{-1} r + \sin\kappa^{-1} r_q \right] \phi_{,\eta} \end{aligned}$$

⁶ Some authors reserve this term for $a_o r$ which is a proper distance on the $\eta = \eta_o$ hypersurface. r here is a dimensionless coordinate distance.

$$\begin{aligned}
& +2r_q^{-1} \int_0^{r_q} \frac{dr}{\sqrt{1-\kappa r^2}} \left[-r\sqrt{1-\kappa r_q^2} \right. \\
& \quad \left. +r_q\sqrt{1-\kappa r^2} \right] \phi_{,\eta} \\
& +2r_q^{-1} \int_0^{r_q} \frac{dr}{\sqrt{1-\kappa r^2}} \left[\sqrt{1-\kappa r_q^2} \sqrt{1-\kappa r^2} \right. \\
& \quad \left. -\kappa r r_q \right] \phi \\
& -r_q^{-1} \int_0^{r_q} \frac{r dr}{\sqrt{1-\kappa r^2}} \left[-r\sqrt{1-\kappa r_q^2} \right. \\
& \quad \left. +r_q\sqrt{1-\kappa r^2} \right] [\phi_{,xx} + \phi_{,yy}]
\end{aligned} \tag{20}$$

where

$$I_{ISW} = -2 \int_0^{r_q} \frac{dr}{\sqrt{1-\kappa r^2}} \phi_{,\eta} \tag{21}$$

is the integrated Sachs-Wolfe effect,

$$I_S = -2 \int_0^{r_q} \frac{dr}{\sqrt{1-\kappa r^2}} \phi \tag{22}$$

is the Shapiro effect and

$$\delta\eta = \frac{a_q}{\dot{a}_q} (v_q^z - v_o^z + \phi_o - \phi_q + I_{ISW}) \quad . \tag{23}$$

Because of the way in which we defined \mathcal{Q} , the observable redshift of our source obeys the standard equation $1+z = a_o/a_q$ and many of the standard cosmological equations apply with their usual interpretation. In particular, the Hubble factor at the location of a source of redshift z is given by⁷

$$H = \frac{\dot{a}}{a^2} = H_o E(z) \tag{24}$$

with

$$E(z) = \sqrt{(1+z)^2(1+z\Omega_m) - z(2+z)\Omega_\Lambda} \quad . \tag{25}$$

Additionally, the comoving proper distance to \mathcal{Q} is

$$a_o r_q = \frac{1}{H_o \sqrt{|\Omega_\kappa|}} \sin_\kappa \left(\sqrt{|\Omega_\kappa|} f_1(z) \right) \tag{26}$$

where $f_1(z)$ is the integral (shown in Fig. 1)

$$f_1(z) = \int_0^z \frac{dl}{E(l)} \tag{27}$$

and we have the usual expressions for the density parameters in matter, vacuum energy, and curvature

$$\Omega_m = \frac{8\pi}{3H_o^2} \rho_{m0} \quad \Omega_\Lambda = \frac{\Lambda}{3H_o^2} \quad \Omega_\kappa = -\frac{\kappa}{a_o^2 H_o^2} \tag{28}$$

respectively. As usual, these constants are related by the identity $\Omega_m + \Omega_\Lambda + \Omega_\kappa = 1$ which has the crucial consequence that for $\Omega_\kappa \neq 0$

$$\begin{aligned}
a_o H_o &= \sqrt{|\Omega_\kappa^{-1}|} \\
&= \sqrt{|(1-\Omega_m-\Omega_\Lambda)^{-1}|}
\end{aligned} \tag{29}$$

and hence a_o is expressible in terms of the usual cosmological parameters. This is in contrast to the flat case where there is no meaningful curvature scale available.

⁷ It is important to realize that the numerator here is a conformal time derivative. This accounts for the unusual power in the denominator.

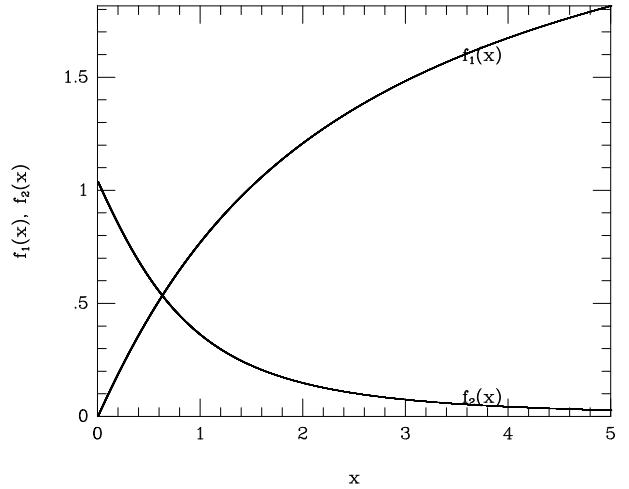
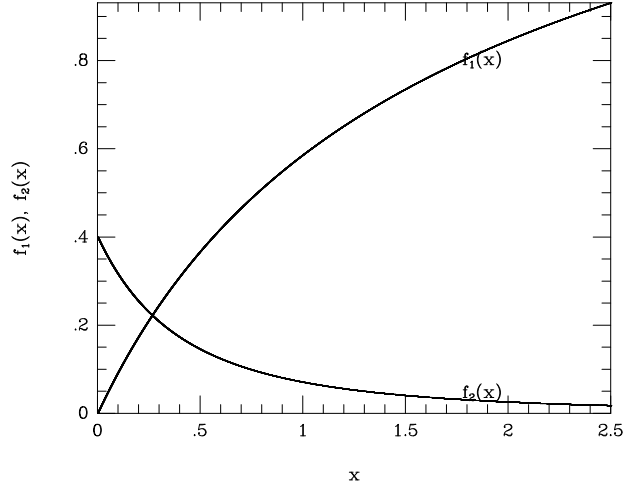


Figure 1. The functions $f_1(x)$ and $f_2(x)$ for $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$ (top) and $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ (bottom).

For the curved cases, then, both r_q and $a_o r_q$ are functions of the standard cosmological parameters and the observed source redshift and so (20) represents the desired formula for the luminosity distance. Simply for clarity we record, for these cases,

$$r_q = \sin_\kappa \left(\sqrt{|\Omega_\kappa|} f_1(z) \right) \tag{30}$$

and, valid for all cases, curved and flat,

$$r_q^{-1} \delta\eta = \frac{\sqrt{|\Omega_\kappa|} (1+z) (v_q^z - v_o^z + \phi_o - \phi_q + I_{ISW})}{E(z) \sin_\kappa \left(\sqrt{|\Omega_\kappa|} f_1(z) \right)} \tag{31}$$

In the flat case our formula simplifies considerably, taking the form

$$\begin{aligned}
\delta &= (v_q^z - \phi_o - 2\phi_q + I_{ISW}) \\
&\quad - r_q^{-1} \delta\eta - 2r_q^{-1} I_S \\
&\quad - r_q^{-1} \int_0^{r_q} dr [-r + r_q] r [\phi_{,xx} + \phi_{,yy}]
\end{aligned} \tag{32}$$

In contrast to the curved cases, in the flat case it is not possible to express r_q in terms of standard cosmological parameters and source redshift alone. As a result, it looks at first as if the terms in (32) involving integrations will prove problematic to our program. In fact, however, this difficulty is easily overcome by the change of variables $r \mapsto r_q w$ and $k = |\mathbf{k}| \mapsto v/r_q$ performed after Fourier decomposition of ϕ with mode-variable \mathbf{k} . This procedure is illustrated in the examples below.

3 LUMINOSITY DISTANCE SCATTER FROM WEAK LENSING

The last term in (32), and its antecedent in (20), are precisely the standard cosmological weak lensing terms usually analyzed in isolation. In this section we illustrate how these terms can be expressed as functions of the standard cosmological parameters, and work out the contribution of the last term in (32) numerically for a simple $\Omega_m = 1$ flat cosmology, and for the currently popular flat $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ cosmology. The identification and analysis of the weak lensing term using a perturbed geodesic equation is not novel (Bernadeau, Van Waerbeke & Mellier 1997; Kaiser 1998). Its emergence as the dominant term in a physically sensible and complete description of the luminosity distance is, however, and we include its analysis here for completeness as it is somewhat tricky to cast in the proper form.

In order to evaluate

$$\delta_{\text{lens}} = \int_0^{r_q} dr \left(\frac{r}{r_q} - 1 \right) r [\phi_{,xx} + \phi_{,yy}] \quad (33)$$

in either cosmological model it is necessary to specify the potential perturbation, ϕ . ϕ satisfies the perturbed Einstein's equations in the longitudinal gauge as given, for example, as equations (5.17)-(5.19) in Mukhanov et al. (1992) for $\Omega_\Lambda = 0$ cosmologies. For the flat $\Omega_m = 1$ cosmology during an epoch of matter-domination, and when the hydrodynamical perturbations described by ϕ are pressureless, ϕ is independent of conformal time, taking the form $\phi(\mathbf{x}, \eta) = \phi(\mathbf{x})$.⁸ For models with vacuum energy, ϕ varies with conformal time according to $\phi(\mathbf{x}, \eta) = s(\eta)\phi(\mathbf{x})$ with $s(\eta)$ obtained from

$$s(z) = (1+z) E(z) \frac{f_2(z)}{f_2(0)} \quad (34)$$

where $f_2(z)$ is the integral

$$f_2(z) = \int_z^\infty \frac{(1+l) dl}{E(l)^3}, \quad (35)$$

(see Fig. 1) and $z(\eta)$ is given through inversion of

$$f_1(z) = a_o H_0 (\eta_o - \eta) \quad (36)$$

The factor $f_2(0)$ in (34) ensures the conventional normalization for the $\Omega_m = 1$ case.

While the temporal variation of the potential is deterministic in the models considered, the spatial variation is unconstrained. It is common to imagine the actual $\phi(\mathbf{x})$ as one

realization drawn from a Gaussian random field. In this picture, the random nature of ϕ is usually expressed by specifying the probability distributions of its Fourier components,

$$\phi(\mathbf{x}, \eta) = s(\eta) \int_V h(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d^3 \mathbf{k}}{(2\pi)^3} \quad (37)$$

with

$$\langle h(\mathbf{k}) h^*(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P(k) \quad (38)$$

$$\langle h(\mathbf{k}) h(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P(k) \quad (39)$$

$$\langle h^*(\mathbf{k}) h^*(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P(k) \quad (40)$$

where the last two correlators follow from the first because of the requirement that ϕ be real.

Now we are ready to compute the RMS fractional deviation due to lensing. It is easiest to take the various correlators into account by writing ϕ in the form

$$\begin{aligned} \phi(\mathbf{x}, \eta) &= \frac{1}{2} \phi(\mathbf{x}, \eta) + \frac{1}{2} \phi(\mathbf{x}, \eta)^* \\ &= \frac{s(\eta)}{2} \int_V h(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d^3 \mathbf{k}}{(2\pi)^3} \\ &\quad + \frac{s(\eta)}{2} \int_V h^*(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{d^3 \mathbf{k}}{(2\pi)^3} \end{aligned} \quad (41)$$

whence

$$\begin{aligned} \langle \delta_{\text{lens}}^2 \rangle &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{k} P(k) [k_x^2 + k_y^2]^2 \\ &\quad \times \left| \int_0^{r_q} dr \left(\frac{r}{r_q} - 1 \right) r s(r) e^{ik_z r} \right|^2 \end{aligned} \quad (42)$$

with $s(r)$ given by the compositions of (34) and (36) above with the identity, true along $x^{(0)}$ where we need it, $(\eta_o - \eta) = r$.

3.1 Lensing effects from a simple power spectrum

We must know the potential power spectrum $P(k)$ to perform the integral in (42). For the purposes of illustration, we follow Seljak (1994) and assume a broken power-law form

$$P(k) = \begin{cases} Ak^{-3} & \text{for } k \leq k_0 \\ Ak_0^4 k^{-7} & \text{for } k \geq k_0 \end{cases} \quad (43)$$

with k_0 , the turnover wavenumber, corresponding to a physical scale of about 10 Mpc. The constant A is set by the CMBR quadrupole of $Q_2^2 = (6 \times 10^{-6})^2$ by

$$Q_2^2 = \frac{20\pi K_2^2}{9} \int_0^\infty k^2 P(k) j_2^2(2k/H_0) dk \quad (44)$$

(Bond & Efstathiou 1987). Here $j_2(x)$ is the spherical Bessel function of order 2 and K_2^2 is the amplification coefficient of Kofman & Starobinskii (1985) which accounts for the time variation of the potential. For the $\Omega_m = 1$ case, $K_2 = 1$ and for the $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ case $K_2 = 1.19$. The normalization for the power spectrum then becomes

$$A = \begin{cases} 6.2 \times 10^{-11} & \text{if } (\Omega_m, \Omega_\Lambda) = (1, 0) \\ 4.4 \times 10^{-11} & \text{if } (\Omega_m, \Omega_\Lambda) = (0.3, 0.7). \end{cases} \quad (45)$$

We simplify the notation by generalizing (43) to

$$P(k) = Ak_0^{n-3} k^{-n} \quad (46)$$

⁸ It is possible to express the Laplacian of $\phi(\mathbf{x})$ in terms of the fractional perturbation to the hydrodynamical energy density on a surface of constant conformal time via what is usually termed a ‘‘cosmological Poisson equation’’ but this is not necessary here.

which can be used both above and below the turnover wavenumber, and change variables in the radial integration to $w = r/r_q$ and in the $|\mathbf{k}|$ integration to $v = kr_q$. Inserting this into (42), we obtain

$$\langle \delta_{\text{lens}}^2 \rangle = \frac{A(k_0 r_q)^{n-3}}{(2\pi)^2} \int dv v^{6-n} \int_{-1}^1 du (1-u^2)^2 \quad (47)$$

$$\times \left| \int_0^1 dw w (w-1) s(w, z) e^{iuvw} \right|^2 \quad (48)$$

for the contributions to $\langle \delta_{\text{lens}}^2 \rangle$ at large and small k . In this equation, $s(w, z)$ given by the composition of $s(\tilde{z})$, given in equation (34), with $\tilde{z}(w, z)$, obtained by solving

$$wf_1(z) = f_1(\tilde{z}) \quad (49)$$

It may be worth noting here that \tilde{z} is simply an intermediate variable that will disappear from the final result whereas z is the observable redshift of the source.

The source redshift and the cosmological parameters Ω_m and Ω_Λ can be seen to enter into $\langle \delta_{\text{lens}}^2 \rangle$ in the same manner: that is, via $s(w, z)$ and the product $v_0 \equiv k_0 r_q$ which is equal to 2π times the comoving proper distance to \mathcal{Q} in units of the turnover wavelength

$$\begin{aligned} v_0 \equiv k_0 r_q &= \frac{2\pi}{\lambda_0 H_0} \int_0^z \frac{dl}{E(l)} \\ &= 1884 h^{-1} f_1(z) \quad (50) \end{aligned}$$

If we use the values of n from (43), then the total luminosity distance scatter $\langle \delta_{\text{lens}}^2 \rangle$ is given by the explicit form

$$\begin{aligned} \langle \delta_{\text{lens}}^2 \rangle &= \frac{A}{4\pi^2} \int_0^{v_0} dv v^3 \int_{-1}^{+1} du (1-u^2)^2 \\ &\quad \times \int_0^1 dw w (w-1) \int_0^1 dw' w' (w'-1) \\ &\quad \times s(w, z) s(w', z) \cos(uv(w-w')) \\ &+ \frac{A}{4\pi^2} v_0^4 \int_{v_0}^\infty \frac{dv}{v} \int_{-1}^{+1} du (1-u^2)^2 \\ &\quad \times \int_0^1 dw w (w-1) \int_0^1 dw' w' (w'-1) \\ &\quad \times s(w, z) s(w', z) \cos(uv(w-w')) \quad (51) \end{aligned}$$

While the calculation of the functions f_1 and f_2 (Fig 1), \tilde{z} (Fig. 2), and s (Fig. 3) present no difficulties, the four-dimensional integrals in (51) have some unpleasant characteristics, which can be brought out by further analysis.

It is convenient to change the order of integration in (51), starting by performing the u -integral, which can be done analytically. Define

$$\begin{aligned} g(x) &\equiv \int_{-1}^{+1} du (1-u^2)^2 \cos(ux) \\ &= -\frac{48}{x^4} \cos x + \frac{48}{x^5} \sin x - \frac{16}{x^3} \sin x \quad (52) \end{aligned}$$

Then integral (51) can be rewritten

$$\begin{aligned} \langle \delta_{\text{lens}}^2 \rangle &= \frac{A}{4\pi^2} v_0^4 \int_0^1 dw w (w-1) s(w, z) \\ &\quad \times \int_0^1 dw' w' (w'-1) s(w', z) t(q) \quad (53) \end{aligned}$$

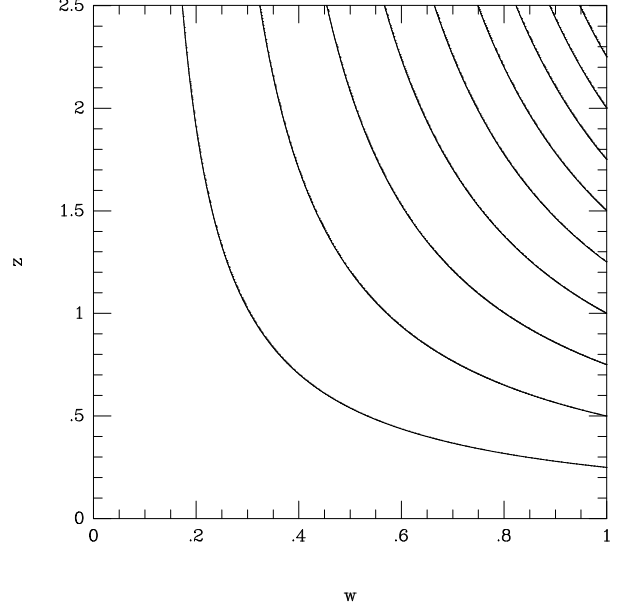


Figure 2. The function $\tilde{z}(z, w)$ for $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$. Contours are drawn at $\tilde{z} = 0.5, 1.0, \dots, 4.5$.

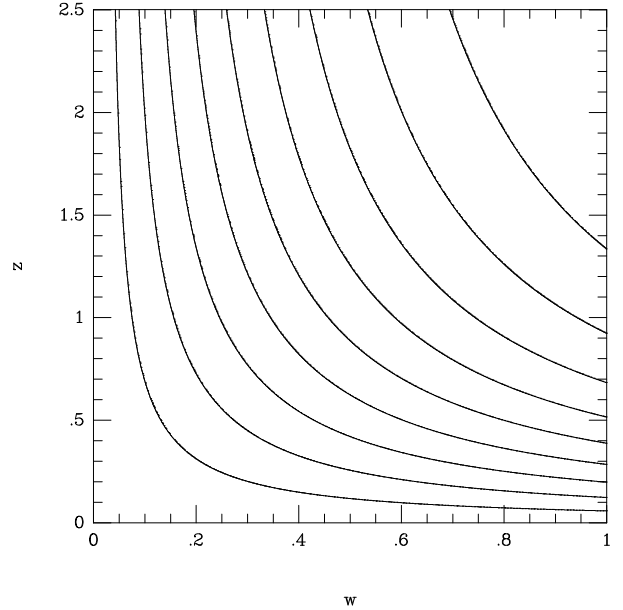


Figure 3. The function $s(z, w)$ for $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$. For $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$, $s = \frac{2}{5}$ for all redshift. Contours are drawn at $s = 1.07, 1.10, \dots, 1.29$.

where the function

$$\begin{aligned} t(q) &= \int_0^1 dy y^3 g(yq) + \int_1^\infty \frac{dy}{y} g(yq) \\ &= \int_0^1 dy \left(y^3 g(yq) + \frac{1}{y} g\left(\frac{q}{y}\right) \right) \quad (54) \end{aligned}$$

and $q = v_0 (w - w')$. $t(q)$ is the key function in what follows, but once calculated, the same $t(q)$ function can be used for all flat cosmologies, since the cosmological behaviour is concealed in the scaling quantity v_0 and in the (w, w') integra-

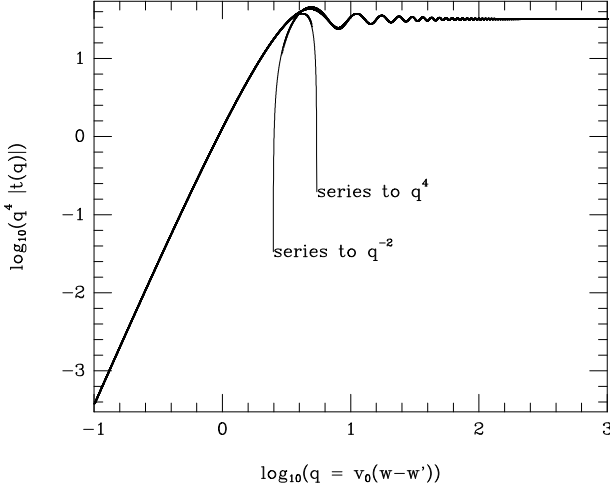


Figure 4. The function $q^4 t(q)$ for $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ and a range of q appropriate to the integral for redshift $z = 2.5$. Note the large range of $t(q)$, the singularity as $q \rightarrow 0$, and the oscillations in $t(q)$ at moderate and large q . This weighted $q^4 t(q)$ has a maximum at $q = 4.77$: $t(q)$ diverges as $-\ln q$ at small q .

tions via the $s(w, z)$ function. Fortunately, the integral in (54) can be performed, to yield

$$t(q) = \frac{16}{15} q^{-5} (-36 \sin q + 6q \cos q + 30q - 2q^2 \sin q - q^3 \cos q + q^4 \sin q - q^5 \text{Ci}(q)) \quad (55)$$

where the reader is to understand $|q|$ for q (formally, the equation is slightly different for negative q , since $t(q)$ is even in q but $\text{Ci}(q)$ is defined with a cut along the negative q axis). The cosine integral function $\text{Ci}(q)$ appearing in (55) causes the major difficulty in evaluating the integral (53) because as $q \rightarrow 0$,

$$t(q) \rightarrow \frac{16}{15} \left(\frac{107}{60} - \gamma - \ln q + \frac{1}{42} q^2 - \frac{1}{4032} q^4 + O(q^6) \right) \quad (56)$$

where $\gamma = 0.577 \dots$ is Euler's constant. This exposes the logarithmic singularity at $q = 0$. By contrast, $\text{Ci}(q)$ is well-behaved as $q \rightarrow \infty$, with $|\text{Ci}(q)| \propto q^{-1}$, so that in this limit

$$t(q) \rightarrow 32 q^{-4} \left(1 - \frac{2}{q} \sin q + \frac{4}{q^2} \cos q + O(q^{-3}) \right) \quad (57)$$

The form of $t(q)$ is shown in Fig. 4: note that the asymptotic forms (56,57) are good descriptions of the behaviour of $t(q)$, but that the logarithmic divergence at $q \rightarrow 0$ makes the integration in (53) improper, with implications for the numerical scheme adopted.

3.2 Numerical results

If we use (43) for the potential power spectrum, then the calculation of the integral for $\langle \delta_{\text{lens}}^2 \rangle$ becomes a problem of covering the (w, w') plane adequately at $w \approx w'$ so that the logarithmic singularity and the oscillations of $t(q)$ for moderate $w - w'$ are well sampled. An efficient solution is to alter the integration variables (w, w') to (w_a, w_b) , where

$$w_a = \frac{1}{\sqrt{2}} (w - w') \quad (58)$$

$$w_b = \frac{1}{\sqrt{2}} (1 - w - w') \quad (59)$$

Since $q = v_0 w_a$, and is independent of w_b , the integration now needs to be done carefully only in the w_a direction, while the change in the integrand in (53) in the w_b direction is slow. Furthermore, the symmetry $(w, w') \rightarrow (w', w)$ implies that only the triangle

$$w_a \in \left[0, \frac{1}{\sqrt{2}} \right] \quad (60)$$

$$w_b \in \left[-\left(\frac{1}{\sqrt{2}} - w_a \right), \left(\frac{1}{\sqrt{2}} - w_a \right) \right] \quad (61)$$

needs to be included in the integral since the $w_a < 0$ triangle gives the same contribution. For the case $(\Omega_m, \Omega_\Lambda) = (1, 0)$, a further simplification is possible, and the smaller triangle

$$w_a \in \left[0, \frac{1}{\sqrt{2}} \right] \quad (62)$$

$$w_b \in \left[0, \left(\frac{1}{\sqrt{2}} - w_a \right) \right] \quad (63)$$

is all that is needed.

The dashed line on Fig. 5 shows the result obtained for $(\Omega_m, \Omega_\Lambda) = (1, 0)$. The effect rises approximately as

$$\sqrt{\langle \delta_{\text{lens}}^2 \rangle} = 12 \times 10^{-2} z^{\frac{3}{2}} \quad (64)$$

at small z (with $h = 0.72$), and reaches about 6% at $z = 1$, and more than 10% at $z = 2$. The solid line on Fig. 5 shows the result for $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$. The curve has a similar shape to that for the Einstein-deSitter case, but larger lensing effects are seen at $z > 1.7$. At $z < 0.4$ this curve can be represented to an accuracy of 4% by

$$\sqrt{\langle \delta_{\text{lens}}^2 \rangle} = 7.5 \times 10^{-2} z^{\frac{3}{2}} \quad (65)$$

The results for both cosmological models show the $z^{\frac{3}{2}}$ dependence that one would have expected on the basis of (51) alone. Numerical results for both cases (accurate to better than 1%) are given in Table 1.

4 SUMMARY

We have presented a complete formula (20) for the luminosity distance in linearly perturbed FLRW spacetimes. The simpler form (32) is appropriate in flat spacetimes. These results give the first explicit presentations of the various gravitational effects which modulate the unperturbed luminosity distance in terms that can be related to observable quantities, as shown explicitly for $(\Omega_m, \Omega_\Lambda) = (1, 0)$ and $(0.3, 0.7)$ cosmologies in Section 3.1.

The cosmological weak lensing term from (32) leads to a fractional scatter in the luminosity distance, $\langle \delta_{\text{lens}}^2 \rangle$, which can be significant as shown in Table 1 and Fig. 5. Clearly this effect is appreciable for quasars and high-redshift galaxies, especially as such objects are now being seen to $z > 6$. The principal assumptions required to derive these results are that the form (43) is a good description of the potential power spectrum, and that the time evolution is accurately described by $s(\eta)$.

At $z = 1$, the value of $\sqrt{\langle \delta_{\text{lens}}^2 \rangle}$ is roughly 0.06. Objects of fixed absolute magnitude would therefore show an

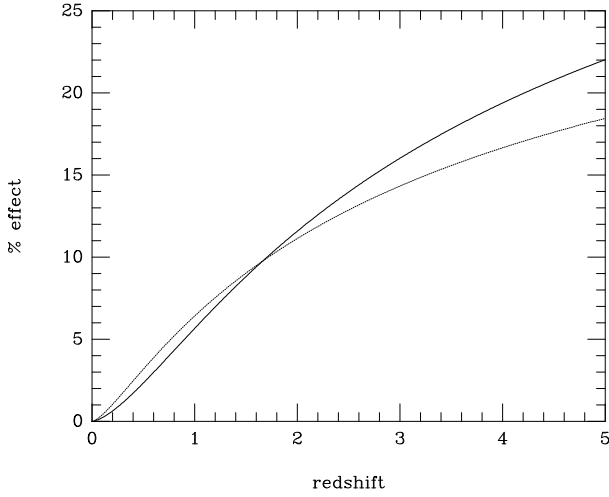


Figure 5. The value of $\sqrt{\langle\delta_{\text{lens}}^2\rangle}$, expressed as a percentage change, as a function of redshift to $z = 2.5$ for $(\Omega_m, \Omega_\Lambda) = (1, 0)$ (dashed line) and $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ (solid line).

apparent magnitude scatter of about 0.12 mag from lensing alone. Since the scatter of SN Ia absolute magnitudes (after correction to a common light curve) is about 0.17 mag (Perlmutter et al. 1999), it is clear that lensing makes a significant, and increasingly important, contribution to the scatter of the SN Ia Hubble diagram as it is extended to redshifts > 1 . At the redshift limit of the SNAP mission ($z \sim 1.7$; Perlmutter et al. 2003), the lensing-induced scatter of supernova apparent magnitudes rises about 0.2 mag, and becomes a dominant contribution to the intrinsic noise in the Hubble diagram. The associated Malmquist bias may also become important.

While the usual cosmological weak lensing term is dominant, other terms are present and can be expected to affect the correlations between luminosity distance corrections and other physical quantities such as the integrated Sachs-Wolfe effect. The approach we have used, direct integration of the null geodesic equation, can also be used to examine the effects of vector and tensor perturbations, and can be extended, with somewhat more difficulty, to higher orders using the results of Pyne & Carroll (1996).

5 ACKNOWLEDGEMENTS

T.P. would like to thank Alex Barnett for substantial help with numerical studies that informed an early draft of the current paper.

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Table 1. Results for $\sqrt{\langle\delta_{\text{lens}}^2\rangle}$ for the two cosmologies

Redshift, z	$\sqrt{\langle\delta_{\text{lens}}^2\rangle}$ (%) for $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$	$\sqrt{\langle\delta_{\text{lens}}^2\rangle}$ (%) for $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$
0.01	0.014	0.007
0.02	0.040	0.021
0.03	0.073	0.038
0.04	0.11	0.059
0.05	0.15	0.082
0.06	0.20	0.11
0.07	0.25	0.14
0.08	0.30	0.17
0.09	0.35	0.20
0.10	0.41	0.23
0.20	1.05	0.64
0.30	1.77	1.16
0.40	2.50	1.74
0.50	3.22	2.37
0.60	3.93	3.02
0.70	4.61	3.70
0.80	5.26	4.37
0.90	5.89	5.05
1.00	6.49	5.73
1.20	7.62	7.04
1.40	8.64	8.31
1.60	9.59	9.51
1.80	10.5	10.6
2.00	11.3	11.7
2.50	13.0	14.0
3.00	14.5	16.2
3.50	15.7	18.0
4.00	16.8	19.6
4.50	17.8	21.0
5.00	18.6	22.3

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